that although for $C = 1.0$, $\ln C/C_0 = 0.133$ and thus the requirement that $C/C_0 \gg 1$ is not fully satisfied, the agreement between the analytical and numerical results is quite satisfactory. Also note that the numerical solutions show a small but finite nondimensional heat-transfer rate, $g'(0)$ at the surface which approaches zero as $\beta \to 0^+$ and $f''(0) \to 0^+$. In view of these results, Kassoy's asymptotic theory is considered confirmed.

Examination of the nondimensional velocity and shearing stress profiles $f'(\eta)$ and $f''(\eta)$ for the compressible boundary-layer case shows that their behavior is similar to that for the incompressible similar boundary layer. Examination of the numerical results for the nondimensional enthalpy ratio $g(\eta)$ shows that the extent of the region of constant enthalpy near the surface increases as $\beta \to 0^+$ and $f''(0) \to 0^+$. In the limit the constant enthalpy region would become of infinite extent. Consequently no heat transfer would occur at the surface. Further examination of the numerical results shows that the region of maximum heat transfer rate moves away from the surface as $\beta \to 0^+$ and $f'(0) \to 0^+$. These results serve to further confirm the asymptotic solutions obtained by Kassoy.

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Laminar Flamesheet with Hypersonic Viscous Interaction

J. W. Ellinwood* and H. Mirels†

The Aerospace Corporation, El Segundo, Calif.

Introduction

Although analytical studies of laminar flamesheets are available,[1] none has considered the case where the flame is held in a hypersonic stream and the heat release is sufficiently intense to induce a significant pressure increase. Evaluation of this induced pressure is the objective of this Note. For simplicity in other respects, we assume gases with unit Prandtl and Lewis number, linear viscosity-temperature variation, and uniform specific heat for all species. The flow geometry assumed is that of Fig. 1, in which the pressure interaction occurs over a semi-infinite, porous, flat plate with fuel injected through it at a specified angle, and the flamesheet is infinitesimally thin. Only self-similar flows are considered. This requires an injection distribution diminishing downstream as $x^{-1/2}$ or $x^{-3/4}$, for weak or strong interaction cases, respectively, and an isothermal wall.

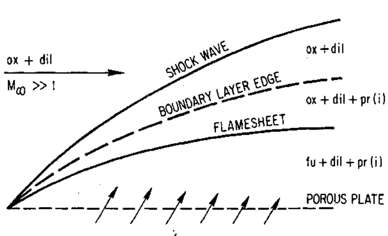

Fig. 1 Self-similar flow model describing pressure interaction between hypersonic flow and laminar flamesheet.

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Index categories: Jets, Wakes, and Viscid-Inviscid Flow Interactions; Combustion in Gases.

* Member of the Technical Staff. Associate Fellow AIAA.

† Head, Aerodynamic and Heat Transfer Department, Laboratory Operations. Associate Fellow AIAA.

Analysis

The self-similar equations of motion for the boundary layer are

$$f''' + ff'' + \beta(g - f'^2) = 0 \quad \text{(momentum)} \tag{1}$$

$$g'' + fg' = 0 \quad \text{(energy)} \tag{2}$$

$$p = \gamma^{-1}(\gamma - 1)\rho h \quad \text{(state)} \tag{3}$$

$$C_i'' + fC_i' = 0 \quad \text{(species)} \tag{4}$$

where

$$f = (2\xi)^{-1/2}\psi, \qquad g = h/H_e \tag{5a, b}$$

$$p = p_\infty P, \qquad \Sigma_i C_i = 1 \tag{6a, b}$$

$$\eta = u_e(2\xi)^{-1/2}\int_0^y \rho\, dy, \qquad \xi = \int_0^x \rho_e \mu_e u_e\, dx \tag{7a, b}$$

$$\beta = (2\xi/u_e)(du_e/d\xi)H_e/h_e \tag{8}$$

Primed quantities denote differentiation with respect to the lone independent variable η. The symbols p, γ, ρ, h, ψ, H, μ, and u represent the gas pressure, specific heat ratio, density, enthalpy, stream function, stagnation enthalpy, viscosity, and x component of velocity, respectively. C_i is the mass fraction of species i. These equations apply between the plate and the flamesheet and between the flamesheet and the outer edge of the boundary layer. Boundary conditions must be applied at the plate, the flamesheet, and the outer edge. At the plate ($y = 0$)

$$-f_w = N\left(2x^{-1}\int_0^x P\, dx\right)^{1/2} \Big/ P \tag{9}$$

$$f_w' = g_w N\bar{\chi}/(M_\infty P \tan\theta) \tag{10}$$

$$C_{i,w}' + f_w C_{i,w} = 0 \qquad (i \neq \text{fu}) \tag{11a}$$

$$C_{\text{fu},w}' + f_w C_{\text{fu},w} = f_w \tag{11b}$$

and g_w is given, where

$$N = \rho_w v_w (\rho_\infty u_\infty)^{-1} Re_\infty^{\ 1/2} \tag{12}$$

$$Re_\infty = \rho_\infty u_\infty x/\mu_\infty, \qquad \bar{\chi} = \tfrac{1}{2}(\gamma - 1)M_\infty^{\ 3} Re_\infty^{\ -1/2} \tag{13a, b}$$

and v_w is the transverse component of velocity at the surface and M is the Mach number. At the flamesheet, f, f', f'', g, C_{dil}, and $C_{\text{pr}(i)}$ are continuous, where subscripts fu, dil, pr (i), and later ox denote fuel, diluent, the ith product, and oxidizer, respectively (see Fig. 1). In addition,

$$g'(\eta_*^{\ +}) = g'(\eta_*^{\ -}) - g_C C_{\text{ox}}'(\eta_*^{\ +})/C_{\text{ox}}(\infty) \tag{14}$$

$$C_{\text{ox}}(\eta_*^{\ +}) = C_{\text{fu}}(\eta_*^{\ -}) = 0 \tag{15}$$

$$v_{\text{ox}} W_{\text{ox}} C_{\text{fu}}'(\eta_*^{\ +}) + v_{\text{fu}} W_{\text{fu}} C_{\text{ox}}'(\eta_*^{\ -}) = 0 \tag{16}$$

where W is the molecular weight and v the stoichiometric coefficients of the following chemical reaction:

$$v_{\text{fu}}[\text{fu}] + v_{\text{ox}}[\text{ox}] \to \Sigma_i v_i [\text{pr}(i)] + \Delta H \tag{17}$$

$$g_C = \Delta H C_{\text{ox}}(\infty)/(v_{\text{ox}} W_{\text{ox}} H_\infty) \tag{18}$$

Physically, g_C is the ratio of combustion energy per unit freestream mass to freestream stagnation enthalpy per unit mass (H_∞). At the outer edge ($\eta \to \infty$), $C_{\text{ox}}(\infty)$ is a specified constant and

$$f'(\infty) = g(\infty) = 1 \tag{19}$$

$$C_{\text{pr}(i)}(\infty) = 0 \tag{20}$$

$$P = \begin{cases} 1 + (\gamma/2)M_\infty \delta^*/x & \text{(weak interaction)} \\ (kM_\infty \delta^*/x)^2 & \text{(strong interaction)} \end{cases} \tag{21a, b}$$

where k is 0.6124, 1.0580, and 1.3201 for $\gamma = 1, \frac{7}{5}$, and $\frac{5}{3}$, respectively.[2] The last relation, for $P(\delta^*)$, is obtained[3] by integrating the equations of motion in the region of disturbed flow outside the boundary layer. Of course, P varies as $x^{-1/2}$ for strong viscous interaction with a flamesheet, just as in the classical interaction case without a flamesheet. As a result, Eq. (9) can be written as

$$-f_w = \begin{cases} 2^{1/2}N & \text{(weak interaction)} \\ 2N/(kM_\infty \delta^*/x) & \text{(strong interaction)} \end{cases} \tag{22a, b}$$

Solution of some of the above equations and conditions is straightforward. Integration of Eq. (2) and use of boundary conditions (14) and (19) lead to a relation between g_w and g_w':

$$g_w = 1 + \frac{g_C C'_{ox}(\eta_*)}{C_{ox}(\infty)} \int_{\eta_*}^{\infty} \exp\left(-\int_{\eta_*}^{\eta} f\, d\eta_1\right) d\eta - g_w' \int_0^{\infty} \exp\left(-\int_0^{\eta} f\, d\eta_1\right) d\eta \tag{23}$$

Similarly, integration of Eq. (4) and use of conditions (11b, 15a, and 16) provide

$$C'_{ox}(\eta_*) = C_{ox}(\infty) \Big/ \int_{\eta_*}^{\infty} \exp\left(-\int_{\eta_*}^{\eta} f\, d\eta_1\right) d\eta \tag{24}$$

$$C = \frac{C_{ox}(\infty) v_{fu} W_{fu}}{v_{ox} W_{ox}} = \frac{-f_w \int_{\eta_*}^{\infty} \exp\left(-\int_0^{\eta} f\, d\eta_1\right) d\eta}{1 - f_w \int_0^{\eta_*} \exp\left(-\int_0^{\eta} f\, d\eta_1\right) d\eta} \tag{25}$$

For hypersonic flows, the displacement thickness δ^* equals the boundary layer thickness δ (corresponding to where $f' = 1$).[2,4] Integration of the inverse of definition (7a) provides

$$M_\infty \delta^*/x = (\hat{\chi}/P)\left(2x^{-1}\int_0^x P\, dx\right)^{1/2} (1 - g_w)(\phi_1 - \eta_*) + g_C \phi_2 + \int_0^{\infty} (1 - f'^2)\, d\eta \tag{26}$$

where

$$\phi_1 = \left[\int_0^{\infty} \exp\left(-\int_0^{\eta} f\, d\eta_1\right) d\eta\right]^{-1} \times \left[\int_0^{\eta_*}\int_0^{\eta} \exp\left(-\int_0^{\eta_1} f\, d\eta_2\right) d\eta_1\, d\eta - \int_{\eta_*}^{\infty}\int_{\eta}^{\infty} \exp\left(-\int_0^{\eta_1} f\, d\eta_2\right) d\eta_1\, d\eta\right]$$

$$\phi_2 = \phi_1 + \left[\int_{\eta_*}^{\infty} \exp\left(-\int_0^{\eta} f\, d\eta_1\right) d\eta\right]^{-1} \times \int_{\eta_*}^{\infty}\int_{\eta}^{\infty} \exp\left(-\int_0^{\eta_1} f\, d\eta_2\right) d\eta_1\, d\eta$$

Equations (23) and (24) have been combined in deriving Eq. (26), and it is clear that if f is known Eqs. (10 and 21–26) provide a unique solution for the unknowns $f_w, f_w', P, M_\infty \delta^*/x, g_w'$, and η_* in terms of the known dimensionless parameters $N, g_w, \hat{\chi}, M_\infty \tan\theta, \gamma$ (directly and indirectly through k and β), g_C, and C. If f is not known, the problem is well posed, but all variables f, g, C_i, and their derivatives must be evaluated simultaneously. Once the problem is solved, it may be desirable to compute such properties as the flamesheet ordinate y_* or some Stanton numbers:

$$M_\infty y_*/x = (\hat{\chi}/P)\left(2x^{-1}\int_0^x P\, dx\right)^{1/2}\left[(1 - g_w)(\phi_3 - \eta_*) + g_C \phi_3 + \int_0^{\eta_*} (1 - f'^2)\, d\eta\right] \tag{27}$$

$$C_{H,w} = \frac{-q_w}{\rho_\infty u_\infty H_\infty (g_{a,w} - g_w)} = P g_w' \left(x \Big/ 2Re_\infty \int_0^x P\, dx\right)^{1/2} (1 + g_C - g_w)^{-1} \tag{28}$$

$$C_{H,*} = \frac{-(\Delta q)_*}{\rho_\infty u_\infty H_\infty} = P g_C \left(x \Big/ 2Re_\infty \int_0^x P\, dx\right)^{1/2} \times \left[\int_{\eta_*}^{\infty} \exp\left(-\int_{\eta_*}^{\eta} f\, d\eta_1\right) d\eta\right]^{-1} \tag{29}$$

where $g_{a,w}$ is the adiabatic g_w (equal to $1 + g_C$) and

$$\phi_3 = \left[\int_0^{\infty} \exp\left(-\int_0^{\eta} f\, d\eta_1\right)\right]^{-1} \int_0^{\eta_*}\int_0^{\eta} \exp\left(-\int_0^{\eta_1} f\, d\eta_2\right) d\eta_1\, d\eta$$

Flow with Full Slip

It is illustrative to examine the trends of our formulation for the simplest model with a flamesheet. Accordingly, we now make assumptions for the three parameters θ; β, and g_w. We assume that the fuel is injected at that angle θ that produces a unit value for f_w'. We neglect the term multiplied by β in Eq. (1), on the basis that β is at most $\frac{1}{7}$ for diatomic species. In this limit, f is simply $f_w + \eta$. The pertinent quadratures involve complementary error functions. Equation (24) becomes

$$\operatorname{erfc}\left(\frac{f_w + \eta_*}{2^{1/2}}\right) = \frac{C}{1 + C}\left[\frac{1 - (\pi/2)^{1/2} f_w \exp(2^{-1} f_w^2) \operatorname{erfc}(2^{-1/2} f_w)}{-(\pi/2)^{1/2} f_w \exp(2^{-1} f_w^2)}\right] \tag{30}$$

Finally, we consider only the case of an insulated wall, $g_w = 1 + g_C$. In this case, the degree of viscous interaction is proportional to g_C only; i.e., the interaction is caused by energy addition at the flamesheet rather than by both $(1 - g_w)$ and g_C, as the right member of Eq. (26) would suggest. These assumptions, with the help of Eq. (22), reduce Eq. (26) to

$$\phi_4 + 1 = \begin{cases} (2N\hat{\chi}g_C)^{-1} M_\infty \delta^*/x & \text{(weak interaction)} \\ 2N^2 (k\hat{\chi}g_C)^{-1} (-f_w)^{-3} & \text{(strong interaction)} \end{cases} \tag{31}$$

where

$$\phi_4 = \{(\pi/2)^{1/2}(-f_w) \exp[2^{-1}(f_w + \eta_*)^2] \operatorname{erfc}[2^{-1/2}(f_w + \eta_*)]\}^{-1}$$

Similarly

$$y_*/\delta = [\eta_*/(-f_w)](1 + \phi_4)^{-1}$$

Note that there is a maximum concentration ratio C for every mass-addition parameter f_w; namely, C is

$$-(\pi/2)^{1/2} f_w \exp(f_w^2/2) \operatorname{erfc}(f_w 2^{-1/2}),$$

according to Eq. (30). With greater C, the supply of fuel is not

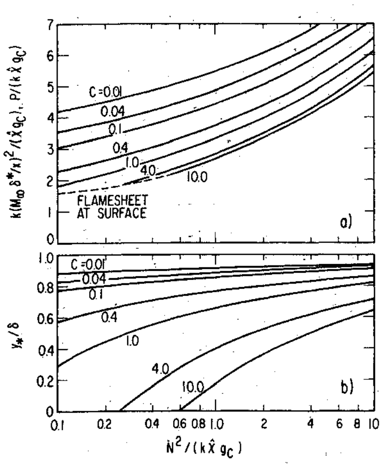

Fig. 2 Flamesheet with strong viscous interaction and full slip: a) boundary-layer thickness and induced pressure; b) flamesheet ordinate.

sufficient to keep the flamesheet from collapsing onto the surface. Inadequately fueled flows can occur in both the weak and strong interaction limits and can be described in terms closely analogous to those for a partly catalytic wall. Surface flamesheet flows are not presented here. In the strong interaction case, C and f_w were assumed, and $(f_w+\eta_*)2^{-1/2}$ was found as the inverse of the complementary error function, Eq. (30). The second essential parameter, $N^2/k\hat{\chi}g_C$, was then found from Eq. (31b). Equation (22b) gives $M\delta^*/x$. Also, $C_{H*}Re_\infty{}^{1/2}/(g_C N) = \phi_4$.

In both the weak and strong interaction limits, the pressure, displacement, energy of combustion, and flamesheet ordinate, which appear in the numerator of the ordinates plotted in Fig. 2, are roughly proportional to the product of the classical interaction parameter $\hat{\chi}$ and the energy parameter g_C. The product $\hat{\chi}g_C$ appears, rather than $\hat{\chi}$ alone, because the characteristic fluid enthalpy in the boundary layer is defined by the combustion process rather than the freestream stagnation enthalpy and surface conduction. Both g_C and g_w would have appeared in a linear form, e.g., Eq. (26), had our application allowed a cooled surface. Note also that injection of a light fuel (small W_{fu} and C) has a larger effecton these properties than a heavy fuel.

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Heat and Mass Transfer on Cones at Angles of Attack

ANDRZEJ WORTMAN*
Northrop Corporation, Hawthorne, Calif.

THIS Note summarizes a parametric study of laminar boundary-layer flows at the most windward generators of sharp cones at angles of attack. Several model gases and equilibrium air flows are considered with Mach numbers, wall to total enthalpy ratios and cross-flows parameters spanning the ranges of main engineering interest. Exact boundary-layer calculations are employed.

Starting with the pioneering work of Moore[1] boundary-layer flow on sharp cones at angles of attack in supersonic flow has received considerable attention as a fluid mechanics rather than a heat-transfer problem. Using exact, constant density-viscosity product solutions with $Pr = 0.7$ for low speed flows and $Pr = 1.0$ for general flows, Reshotko[2] suggested a $Pr^{0.37}$ and $Pr^{0.5}$ variation of heat-transfer and recovery factors, respectively. Some exact solutions,[3,4] based on experimentally determined pressure distributions and normalized by the leading cone generator values were found to be in good agreement with test data. Mass transfer effects on slender cones were studied by Fannelop and Smith,[5] under the assumption of small cross-flow.

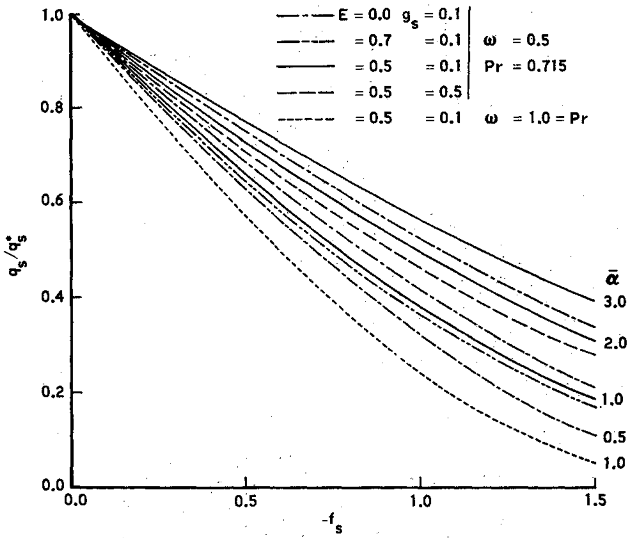

Fig. 1 Reduction of heat transfer by injection.

In a parametric study illustrating the salient features of the flow, Libby[6] employed the assumptions of a constant density-viscosity product and Prandtl number of unity, thus concerning himself primarily with the fluid mechanics aspects of the problem rather than heat transfer which is generally critically dependent on variations of gas properties. In contrast to Libby's work the main interest here is directed at the display of the influence of variation of gas properties.

Table 1 Influence of gas properties on $q_s^*/q_{s,0}$

$\bar{\alpha}$	Pr / ω	0.715 / 0.5	0.715 / 0.7	0.715 / 1.0	1.00 / 1.00	10,000 fps	20,000 fps	25,000 fps
				a) $g_s = 0.10, E_1 = 0.5$				
0.0		1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5		1.30	1.29	1.29	1.30	1.30	1.31	1.30
1.0		1.53	1.53	1.52	1.54	1.54	1.55	1.54
2.0		1.92	1.91	1.90	1.94	1.94	1.94	1.94
3.0		2.24	2.24	2.22	2.26	2.26	2.28	2.26
				b) $g_s = 0.10, E_2 = 0.7$				
0.0		1.00	1.00	1.00	1.00	1.00		1.00
0.5		1.33	1.33	1.33	1.34	1.33		1.33
1.0		1.60	1.59	1.59	1.61	1.59		1.58
2.0		2.02	2.01	2.00	2.04	2.00		1.99
3.0		2.36	2.35	2.34	2.39	2.35		2.34
				c) $g_s = 0.10, E_1 = 0.9$				
0.00		1.00	1.00	1.00	1.00	1.00		
0.5		1.48	1.48	1.48	1.51	1.49		
1.0		1.78	1.81	1.82	1.85	1.80		
2.0		2.29	2.34	2.34	2.38	2.31		
3.0		2.70	2.76	2.76	2.81	2.75		
				c) $g_s = 0.50, E_1 = 0.7$				
0.0		1.00	1.00	1.00	1.00			
0.5		1.41	1.42	1.42	1.42			
1.0		1.71	1.73	1.72	1.72			
2.0		2.19	2.20	2.21	2.19			
3.0		2.58	2.60	2.60	2.58			
				d) $g_s = 0.05, E_1 = 0.7$				
0.0		1.00	1.00	1.00	1.00			
0.5		1.32	1.32	1.32	1.34			
1.0		1.58	1.57	1.57	1.00			
2.0		1.99	1.98	1.97	2.01			
3.0		2.33	2.32	2.31	2.36			

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* Engineering Specialist, Aerodynamics Research Branch; also Post-doctoral Scholar, University of California, Los Angeles; on leave, presently U.S. National Academy of Sciences exchange scientist with the Polish Academy of Sciences, Warsaw, Poland.